Brouwer-Weyl Continuum Through 3D Glasses: Geometry, Computation, General Relativity

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Since at least the 1980s there has been growing interest in the hypothesis that conepts of computability are (or should be) dependent on physics. In the first part of this talk I review some of the fascinating arguments that appear to be at odds with one another more often than one would like.

The guiding idea is that theoretical computational devices like Turing machines ought to be viewed as realized (or realizable) in a particular physical setting. E.g., Turing machines seem to conceptually "live" in the world of classical mechanics. What is meant by "physical setting", however, is in fact a mathematical model of the physical world. Hence it seems more accurate to say that in works on a "physical Church-Turing Thesis", computability is considered in the framework of a *theory* within mathematical physics: classical mechanics, general relativity, or quantum theory, as the case may be.

Although literature on quantum computing features periodic announcements of purported violations of the Church-Turing Thesis, perhaps the most radical expression of the thesis that computability is dependent on physics comes from general relativity. In the somewhat exotic Malament-Hogarth spacetimes a Turing machine can travel along a trajectory that has infinite proper time and, it is argued, can send a signal to an observer in whose frame the machine's trajectory has finite time. The observer would thus have at their disposal an infinite-time Turing machine. Therefore, Hogarth argues [3], "the Church-Turing Thesis is like the outmoded claim: 'Euclidean geometry is the true geometry.'"

The arguments mentioned above proceed in the broader mathematical context of classical analysis, as does most of mathematical physics. In particular, the spacetime continuum is a manifold consisting of points with coordinates in $\mathbb{R}$. In the second part of the talk I would like to add to the overall confusion by showing how the Brouwer-Weyl continuum, or an analogous concept in several dimensions, itself relates to a spacetime familiar from mathematical physics — deSitter spacetime — as well as to order structures introduced in Dana Scott's work on the theory of computation.

According to Brouwer (and, for a time, Weyl), the continuum should be regarded as the collection of 'sequences of nested intervals whose measure converges to zero.' [5]. A higher-dimensional analog would be nested sequences of spheres with radii converging to zero. Classical geometry, going back to Laguerre and Lie, encodes the space of *oriented spheres* in $\mathbb{R}^n$ as points in $\mathbb{R}^{n+1}$: $(\mathbf{x}, r)$ with $\mathbf{x} \in \mathbb{R}^n$ being the centre and $r \in \mathbb{R}$ the oriented radius [1]. In this cyclographic representation the space of spheres has the structure of Minkowski space $\mathbb{R}^{1,n}$ with the usual pseudometric.

In this representation, for $r_1, r_2 > 0$, $||\mathbf{x}_1 - \mathbf{x}_2|| \leq r_1 - r_2$ iff the sphere $(\mathbf{x}_2, r_2)$ is contained in the sphere $(\mathbf{x}_1, r_1)$ [2]. In the terminology of special relativity, sphere inclusion corresponds to events that are related in the causal order. The concept of a nested sequence of spheres thus corresponds to a time-oriented causal sequence of events in Minkowski space.

Restricting to positive radii does not correspond to the full Minkowski space. To deal with this, we consider a different representation, in terms of Lie cycles. For a sphere $(\mathbf{x}, r)$, with $r > 0$, consider the vector $(y_0, \ldots, y_{n+1}) \in \mathbb{R}^{1,n+2}$ given by

$$\begin{aligned} y_0 &= -\frac{1}{2}\Big(\frac{||\mathbf{x}||^2 + 1}{r} - r\Big) \\ (y_1, \ldots, y_n) &= -\frac{1}{r}\mathbf{x} \\ y_{n+1} &= -\frac{1}{2}\Big(\frac{||\mathbf{x}||^2 - 1}{r} - r\Big) \end{aligned}$$

Then $-y_0^2 + \sum_{k=1}^{n+1} y_k^2 = 1$, and inducing the metric on this hyperboloid from $\mathbb{R}^{1,n+2}$ one gets the deSitter metric on the space of spheres

$$ds^2 = \frac{1}{r^2}(-dr^2 + d\mathbf{x}^2).$$

Substitution $r = e^{\mp t}$ yields the deSitter metric $ds^2 = -dt^2 + e^{\pm 2t} d\mathbf{x}^2$ in flat slicing coordinates of the "expanding" (resp. "contracting") part.

Thus, surprisingly, a detour through classical geometries relates the "higher-dimensional continuum" to a well known object in general relativity. Nested sequences of spheres correspond to "time"-oriented causal sequences. Such sequences, without additional qualifications, could be finite; this is not what Brouwer and Weyl had in mind. A more precise analog would be *inextendible* "time"-oriented causal sequences (by analogy of inextendible causal curves): there is no sphere that is contained in all spheres in the nested sequence.

These meditations suffer from a fatal flaw: they invoke the classical analysis that underpins the definition of deSitter space or Lorenzian manifolds in general. Although theorems in general relativity show that the manifold topology (under some assumptions) can be recovered from the space of timelike curves, this requires a notion of smoothness. It may be possible to formulate these ideas in a way that does not presuppose a concept of smoothness. Notably, Martin and Panangaden [4] introduce the category of globally hyperbolic posets, which includes causal orders on globally hyperbolic spacetimes such as deSitter. This category is equivalent to the category of interval domains, introduced by Scott in his pioneering work on the theory of computation and semantics of programming languages.

The upshot of the argument in [4] is that manifold topology (if not geometry) of a globally hyperbolic spacetime can be recovered from a countable dense subset of the associated interval domain of the causal order. The spacetime itself (if we start from one) is homeomorphic to the set of maximal elements in the interval domain, with Scott topology. If no manifold is given from the start, but only a countable dense poset — e.g., spheres with rational centres and rational radii — one can take an ideal completion of the basis of intervals in the poset. The set of maximal elements of the completion, with Scott topology, is the "manifold", topologically; but there is no metric. (This is the fundamental problem of the causal set program.)

Despite interesting and surprising (at least to me) connections with different fields of mathematics, it is not clear whether such an operation, even if successful, would lead to a satisfactory model of the intuitionist continuum in higher dimensions. Automorphisms of the causal order of the Minkowski space $\mathbb{R}^{1,n}$ for $n > 1$ are precisely the Lorenz transformations, by a famous theorem of Alexandrov and Zeeman. In this sense, the structure of the continuum as a set of nested sequences of intervals (which would correspond to $n = 1$) seems to be fundamentally different from a higher-dimensional analog: the Alexandrov-Zeeman theorem does not hold for $n = 1$, as there are nonlinear bijections $\mathbb{R} \to \mathbb{R}$ that preserve interval order.

References

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